# Flux-balance laws in self-force theory

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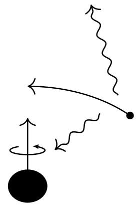
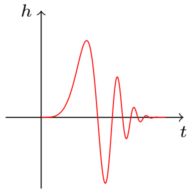
# Flux-balance laws

- Goal of self-force program: radiation during inspiral
- BH perturbation theory: motion $\implies$ radiation
- Motion depends on (regularized) fields at body: these are hard to compute!
- Goal of flux-balance:
  - Derive *some aspects* of motion w/ asymptotic fields (easier to compute)
  - Exploit symmetries of background spacetime: conserved quantities and currents



$\Downarrow \qquad \Uparrow$ (?)



# Scalar self-force

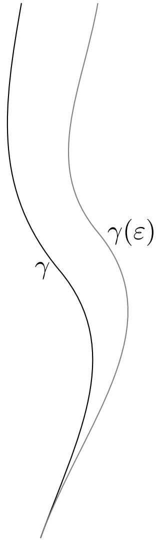
- Restricting to scalar fields for simplicity
- Particle follows $\gamma(\tau; \varepsilon)$, where

$$\begin{gathered}\dot{p}_a(\varepsilon) = -\varepsilon q \nabla_a \phi^{\mathrm{R}} + O(\varepsilon^2), \\ p_a(\varepsilon) \equiv m(\varepsilon) g_{ab} \dot{\gamma}^b(\varepsilon), \quad \dot{\gamma}^a(\varepsilon)\dot{\gamma}^b(\varepsilon) g_{ab} = -1\end{gathered} \tag{1}$$

- Properties of $\phi^{\mathrm{R}}$:
  - Vacuum solution: $\Box \phi^{\mathrm{R}} = 0$
  - Smooth on worldline
  - Depends on the *background* worldline through

$$\rho(x) \equiv q \int_{-\infty}^{\infty} \mathrm{d}\tau' \, \delta[x, \gamma(\tau')]$$

- Goal of this work: solve eqn. (1) using conserved currents



# Outline

I. Background

II. Symmetry operators and symplectic currents

III. Hamiltonian formulation

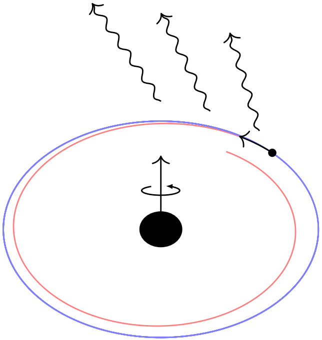
IV. Future work

# Osculating geodesics

- Geodesic motion (in Kerr): four constants $Q$
- Non-geodesic motion: can be represented by "osculating" geodesics, with varying "conserved quantities" $Q(\tau)$
- Self-force directly gives $\mathrm{d}Q/\mathrm{d}\tau$:

$$\frac{\mathrm{d}Q}{\mathrm{d}\tau} = \varepsilon F_Q$$

Is it possible to get $F_Q$ from fluxes?



# Intuitive picture

- Conserved quantities from isometries:

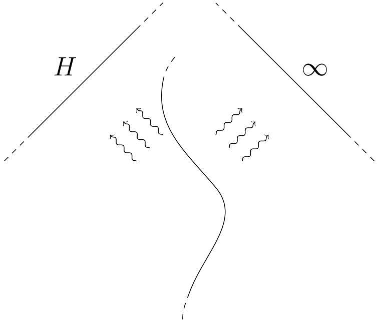
$$\underbrace{E_\xi = \xi^a p_a}_{\text{conserved quantity}} \iff j^a_{\text{part.}} \equiv \underbrace{T^{ab}_{\text{part.}} \xi_b}_{\text{conserved current}}$$

- Particle & field contribute to $T^{ab}$:

$$T^{ab} = T^{ab}_{\text{part.}} + T^{ab}_{\text{field}} \implies \nabla_b T^{ab} = 0$$

- Flux-balance law:

$$\Delta E_\xi \stackrel{?}{=} -\left(\int_H j^a_{\text{field}} \mathrm{d}\Sigma_a + \int_\infty j^a_{\text{field}} \mathrm{d}\Sigma_a\right)$$



# Prior work

- Initial work: [Gal'tsov, 1982] showed balance for "all time":

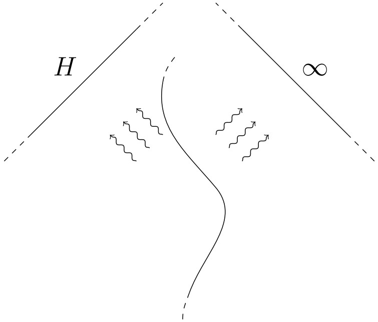
$$\lim_{T\to\infty}[E_\xi(T) - E_\xi(-T)] = -\left(\int_H j^a_{\text{field}} \mathrm{d}\Sigma_a + \int_\infty j^a_{\text{field}} \mathrm{d}\Sigma_a\right)$$

(rigorous derivation: [Quinn & Wald, 1999])

- Orbit averages (e.g., [Mino, 2003]):

$$\langle \Delta E_\xi \rangle = (\text{terms at } H) + (\text{terms at } \infty)$$

- These formulations related by averaging over all time



# Carter constant

- Kerr only has *two* isometries, $t^a$ & $\phi^a$:

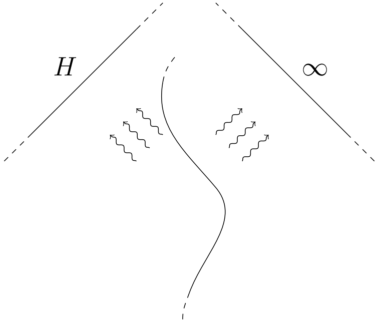
$$E \equiv -t^a p_a, \qquad L_z \equiv \phi^a p_a$$

- Third conserved quantity, the "Carter constant":

$$K \equiv K_{ab} p^a p^b \qquad (K_{ab} \text{ a ``Killing tensor''})$$

- *Cannot* make conserved current w/ $K_{ab}$ & $T^{ab}$ alone [G & Flanagan, 2015]

- $\langle \Delta K \rangle$ involves terms at $H$ & $\infty$ *, despite *no known flux-balance law*—why? (e.g., [Mino, 2003] & [Isoyama et al., 2018])



# Outline

# Non-stress-energy currents

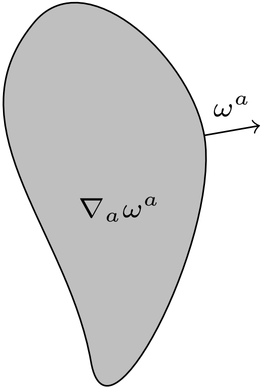
- Perturbations to any field theory $\Phi$ possess conserved, "symplectic" current: $\omega^a[\delta_1\Phi, \delta_2\Phi]$
- (Linear) scalar fields: $\omega^a$ is the Klein-Gordon current:

$$\omega^a[\phi_1, \phi_2] = \phi_1 \nabla^a \phi_2 - \phi_2 \nabla^a \phi_1$$

- Conserved for "vacuum" solutions $\phi_1$, $\phi_2$:

$$\nabla_a \omega^a = \phi_1 \Box \phi_2 - \phi_2 \Box \phi_1$$

- Need two fields!



# Symmetry operators

- Symmetry operator $\mathcal{D}$:

$$\Box \mathcal{D}\phi = \widetilde{\mathcal{D}}\Box\phi \quad \Longrightarrow \quad \mathcal{D} \text{ maps b/w solutions of } \Box\phi = 0$$

- Quadratic conserved current $\omega^a[\phi, \mathcal{D}\phi]$
- Examples:
  - For isometry $\xi^a$,

$$\mathcal{D}_\xi : \phi \mapsto \xi^a \nabla_a \phi$$

  - For Killing tensor $K_{ab}$,

$$\mathcal{D}_K : \phi \mapsto \nabla_a(K^{ab}\nabla_b\phi) \qquad \text{[Carter, 1977]}$$

# Flux-balance laws for isometries

- General symmetry operator, fields:

$$\nabla_a \omega^a[\phi_1, \mathcal{D}\phi_2] = \phi_1 \widetilde{\mathcal{D}} \Box \phi_2 - (\mathcal{D}\phi_2)\Box\phi_1$$

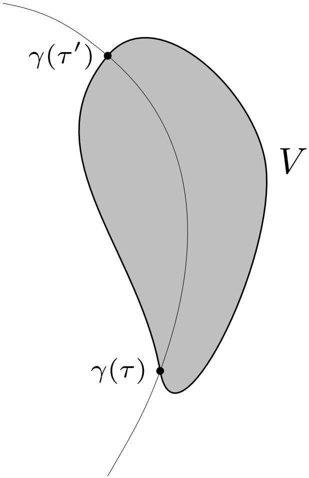
- Change in $E_\xi \equiv \xi^a p_a$ between $\tau$ & $\tau'$:

$$\frac{\mathrm{d}E_\xi}{\mathrm{d}\tau} \simeq -\varepsilon \xi^a \nabla_a \phi^{\mathrm{R}} \implies \Delta E_\xi \simeq -\varepsilon \int_V \rho \mathcal{D}_\xi \phi^{\mathrm{R}}$$

- By Stokes' theorem,

$$\boxed{\Delta E_\xi = \varepsilon \int_{\partial V} \omega[\phi^{\mathrm{ret.}}, \mathcal{D}_\xi \phi^{\mathrm{R}}] + O(\varepsilon^2)}$$

with $\Box\phi^{\mathrm{ret.}} = \rho$



# Issues with the Carter constant

- First-order change in Carter constant:

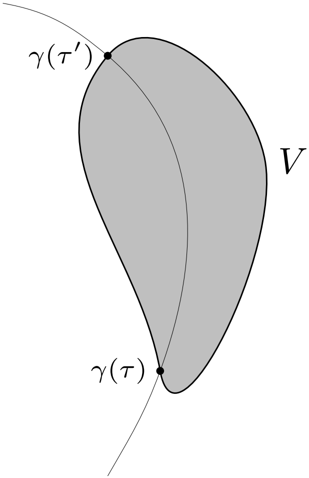
$$\Delta K = -2\varepsilon \int_V \rho K^{ab} p_a \nabla_b \phi^{\mathrm{R}} + O(\varepsilon^2)$$

- Flux of analogous conserved current:

$$2\varepsilon \int_{\partial V} \omega[\phi^{\mathrm{ret.}}, \mathcal{D}_K \phi^{\mathrm{R}}] = -2\varepsilon \int_V \rho \nabla_a (K^{ab} \nabla_b \phi^{\mathrm{R}})$$

$$\stackrel{?}{\neq} \Delta K$$

- Consequence of quadratic nature of $K$...?
- *: $\langle \Delta K \rangle$ "flux-balance" differs from $\langle E \rangle$, $\langle L_z \rangle$: involves functions averaged over the worldline!



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# Self force as a Hamiltonian system

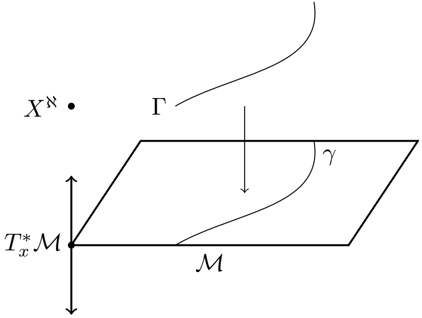
- Points on phase space: $X^{\aleph} = \binom{x^\alpha}{p_\alpha}$
- Scalar self-force has the following Hamiltonian:

$$H(X;\varepsilon) = -\sqrt{-g^{\alpha\beta}(x)p_\alpha p_\beta} + \varepsilon q \phi^{\mathrm{R}}(x) + O(\varepsilon^2)$$

- Trajectory on phase space:

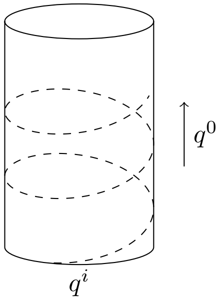
$$\dot{\Gamma}^A(\varepsilon) = \Omega^{BA}(\mathrm{d}H)_B(\varepsilon) + O(\varepsilon^2)$$

where $\underbrace{\boldsymbol{\Omega} \equiv \mathrm{d}p_\alpha \wedge \mathrm{d}x^\alpha}_{\text{“symplectic two-form”}}$, $\Omega^{AC}\Omega_{CB} = \delta^A{}_B$



# Action-angle variables

- Geodesic motion in Kerr is *integrable*
- Can write coordinates $X^{\aleph} = \binom{q^\alpha}{J_\alpha}$ such that
  - $J_\alpha$ are constants of motion
  - $q^0$ is non-compact, $q^1, \cdots, q^3$ periodic in $2\pi$
  - $\boldsymbol{\Omega} = \mathrm{d}J_\alpha \wedge \mathrm{d}q^\alpha$
- Our flux-balance laws give expressions for $\Delta J_\alpha \implies \Delta K$ (numerically)



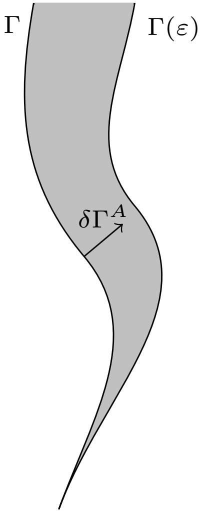

Surface of constant $J_\alpha$

# Perturbation vector

- $\Delta J_\alpha$ isn't covariant: just difference of coordinates
- Instead, consider $\delta\Gamma^A$, the tangent to $\Gamma(\tau;\varepsilon)$ *at constant* $\tau$
- Evolution:

$$\pounds_{\dot{\Gamma}} \delta\Gamma^A = \Omega^{BA} (\mathrm{d}\delta H)_B$$

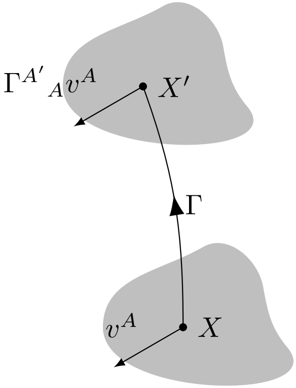
- How to integrate?



# Hamiltonian transport

- Can consider $\Gamma : X \mapsto X'$ for fixed $\tau, \tau'$
- *Pushforward* $(\Gamma^*)^{A'}{}_A$ relates vectors at $X$ & $X'$
- In coordinates: $(\Gamma^*)^{\aleph'}{}_{\aleph} = \partial X^{\aleph'} / \partial X^{\aleph}$
- First-order correction to $\Delta J_\alpha$ a component of

$$\begin{aligned} \Delta\Gamma^A &\equiv (\Gamma^*)^A{}_{A'} \delta\Gamma^{A'} - \delta\Gamma^A \\ &= \Omega^{BA} \int_\tau^{\tau'} \mathrm{d}\tau'' (\Gamma^*)^{B''}{}_B (\mathrm{d}\delta H)_{B''} \end{aligned}$$



# A new symmetry operator

- Fields which we consider are functions of $\Gamma$:

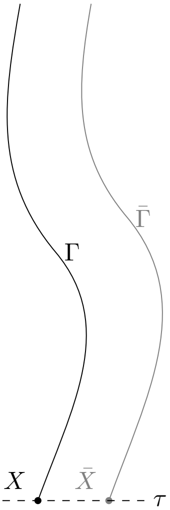
$$\Gamma(\tau) \implies \rho(x) \implies \phi^{\mathrm{R,ret.}}(x)$$

- For fixed $\tau$, $\Gamma$ is a function of its initial data $X$ at $\tau$

- New symmetry operator $\mathcal{D}_A$ varies initial data $X$: e.g.,

$$\mathcal{D}_A \phi^{\mathrm{ret.}}(x', X) = \int G^{\mathrm{ret.}}(x', x'') \mathcal{D}_A \rho(x'', X)$$

$$\implies \Box \mathcal{D}_A \phi^{\mathrm{ret.}} = \mathcal{D}_A \rho$$



# Flux-balance law

- Results for general symmetry operators:

$$\nabla_a \omega^a[\phi^{\mathrm{R}}, \mathcal{D}_A \phi^{\mathrm{ret.}}] = \phi^{\mathrm{R}} \mathcal{D}_A \rho$$

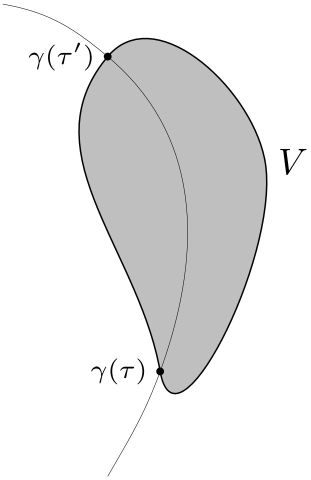
- Can show

$$\Delta\Gamma^A = \Omega^{AB} \int_V \phi^{\mathrm{R}} \mathcal{D}_B \rho,$$

giving flux-balance law:

$$\boxed{\Delta\Gamma^A = \Omega^{AB} \int_{\partial V} \omega[\phi^{\mathrm{R}}, \mathcal{D}_A \phi^{\mathrm{ret.}}]}$$

- Procedure: compute $\phi^{\mathrm{R,ret.}}$ asymptotically & vary w.r.t. parameters of worldline



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# Regions of integration

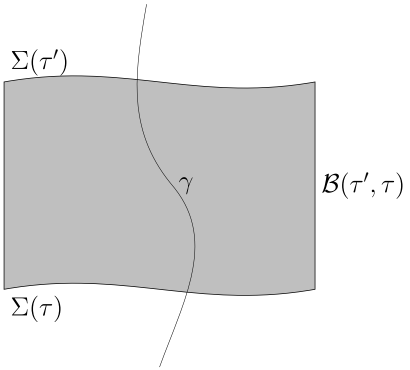
- Results are not simply in terms of "fluxes":

$$\Delta Q = \underbrace{\int_{\Sigma(\tau')} J - \int_{\Sigma(\tau)} J}_{\text{“endcap” contributions}} + \int_{\mathcal{B}(\tau',\tau)} J$$

- Can be understood in terms of "renormalization":

$$\Delta \left[ Q - \int_{\Sigma} J \right] = \int_{\mathcal{B}(\tau',\tau)} J$$

- $\Delta \int_{\Sigma} J$ vanishes over "infinite time average"
  $\implies$ results comparable to prior literature



# Generalization to gravity

Each piece of calculation *should* generalize straightforwardly:

- Symplectic current:
  - Form given by, e.g., [Burnett & Wald, 1990]:

$$\omega^a[\delta_1 g, \delta_2 g] = P^{abcdef}\left[\delta_1 g_{bc} \nabla_d \delta_2 g_{ef} - (1 \longleftrightarrow 2)\right]$$

  - Conserved for vacuum perturbations:

$$\nabla_a \omega^a[\delta_1 g, \delta_2 g] = \delta_1 g_{ab} \underset{(1)}{G}{}^{ab}[\delta_2 g] - (1 \longleftrightarrow 2)$$

  - Gauge invariant up to total derivatives
- Symmetry operators:
  - Isometries $\xi^a$: $\mathcal{D}_\xi = \pounds_\xi$
  - Generalizations of $\mathcal{D}_K$: many, see e.g. [G & Flanagan, 2020]
  - Note: generalizations of $\mathcal{D}_K$ still involve higher derivatives $\implies$ do not give useful flux-balance laws?
- Hamiltonian approach: still in progress, but seems to work

# Second-order self-force

More significant issues arise at second order:

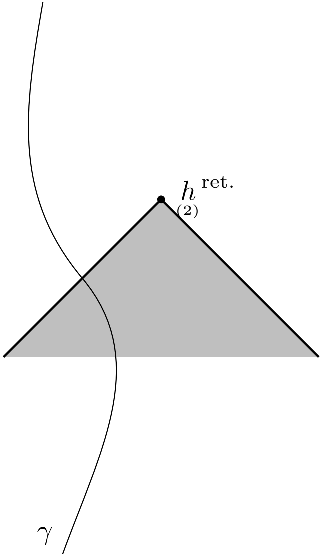
- First-order results depended on $\Box\phi^{\mathrm{R}} = 0$, but analogue at second order fails:

$$\underset{(1)}{G}_{ab}\big[\underset{(2)}{h}{}^{\mathrm{R}}\big] \sim \underset{(2)}{G}_{ab}\big[\underset{(1)}{h}, \underset{(1)}{h}\big]$$

- Equations of motion simply more complicated; may be easier to deal directly with

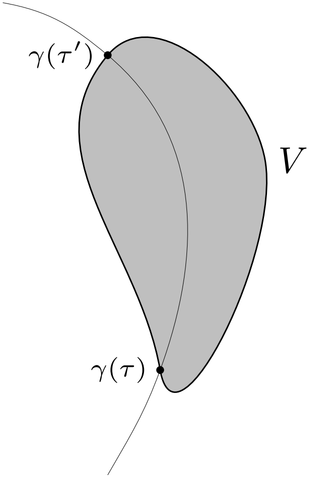
$$h^{\mathrm{R}}_{ab} = \varepsilon \underset{(1)}{h}{}^{\mathrm{R}}_{ab} + \varepsilon^2 \underset{(2)}{h}{}^{\mathrm{R}}_{ab} + O(\varepsilon^3)$$

as $\gamma(\varepsilon)$ is geodesic in $g_{ab} + h^{\mathrm{R}}_{ab}$



# Conclusions

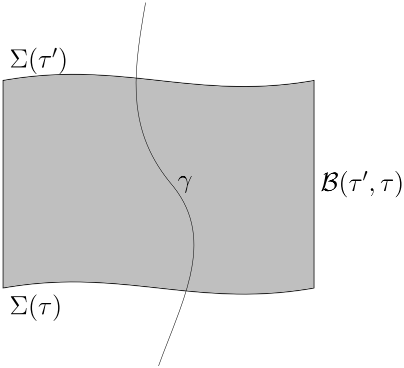
- Two types of flux-balance laws (for scalar fields): conserved currents determine
  - Constants of motion (for spacetime isometries)
  - Phase space evolution $\implies$ *all* constants of motion (including Carter constant!) from action variables
- Flux-balance laws for phase space evolution *require* considering non-stress-energy conserved currents
- Hamiltonian approach may even be useful when background motion is not integrable, e.g. with spins



# Conclusions (cont'd)

- Issues of practicality arise from "endcap" contributions, but *all* flux-balance laws face this

- Results should generalize easily to gravity, but second-order will require further work



# Backup slides

# Self-consistent evolution

- Results gave differences from background geodesic
- More useful are "self-consistent" approaches,
  w/ self-force evolution in terms of exact curve
- Evolution equations may have flux-balance form:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(Q - \int_{\Sigma} J\right) = \int_{\mathcal{B}} J$$